Strictly Pseudo-Regular and Strictly Pseudo-Normal Topological Spaces

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Abstract:In this paper strictly pseudo-regular and strictly pseudo-normal topological spaces have been defined and their properties have been studied. In the former class a compact set can be separated from an external point by a continuous function, while in the latter, two disjointcompact sets can be separated by a continuous function.Many important properties have been proved.

Keywords: Strictly pseudo-regular spaces, strictly pseudo-normal spaces, compact sets, Hausdorff spaces, completely regular spaces, continuous mapping.

Mathematics Subject Classification: 54D10, 54D15, 54A05, 54C08.

I. INTRODUCTION

This is the third in a series of our papers. Here we havedefined andstudied two new classes oftopological spaces.A topological space X will be called strictly pseudo-regular if for each compact set K and for every $x \in X$ with $x \notin K$, there exists a continuous function $f: X \to [0,1]$ such that f(x) = 0 and f(K)=1.X will be called strictly pseudo-normal if for each pair of disjoint compact subsets K_1, K_2 of X, there exists a continuous function $f: X \to [0,1]$ such that f(K_1)=0 and f(K_2)=1. We haveestablished variousproperties ofthesespaces. Thestrictly pseudo-regular spaces resemble thecompletely regular spaces. Many suchproperties hold for these two classes. For Hausdorff spaces, 'completely regular' and'completely normal' are synonymous with'strictly pseudo-regular' and 'strictly pseudo-normal'spaces respectively. For compactspaces 'strictly pseudo-regular'implies'completely regular'and 'strictly pseudo-normal'implies 'completely normal'.Every metric space is both strictly pseudo-regular and strictly pseudo-normal.

The first and the second such papers has appeared in 2018([17],[18]). Earlier, regular and normal topological spaces have been generalized in various other ways. p-regular, p-normal, β-normal and γ-normal spaces ([7], [8], [10], [12], [15]) are several examples of some of these.We have used the terminology and definitions of text books of S. Majumdar and N. Akhter [1], Munkres [2], Dugundji [3], Simmons [4], Kelley [5] and Hocking-Young [6].

We shall now define and study strictly pseudo-regular spaces as a generalization ofcompletely regular spaces.

II. STRICTLY PSEUDO-REGULAR SPACES

Definition 2.1: A topological space X will be called**strictly pseudo-regular** if for each compact set K and for every $x \in X$ with $x \notin K$, there exists a continuous function $f: X \to [0,1]$ such that f(x)=0 and f(K)=1.

Example 2.1: Let K be a compact subset of ℝ and let $x \in \mathbb{R}$ such that $x \notin K$. Since ℝ isHausdorff, K is closed and since ℝ is completely regular,there exists a continuous function $f: X \to [0,1]$ such that f(x)=0 and f(K)=1. Thus ℝ is strictly pseudo-regular.

Theorem 2.1: Every strictly pseudo-regular compact space is completely regular.

Proof: Let X be compact and strictly pseudo-regular. Let K be a closed subset of X and let $x \in X$ with $x \notin K$.Since X is compact, K is compact. Again, since X is strictly pseudo-regular,there exists a continuous function $f: X \to [0,1]$ such that f(x)=0 and f(K)=1. Therefore X is completely regular.

Theorem2.2: Every completely regular Hausdorff space is strictly pseudo-regular.

Proof: Let X be a completely regular Hausdorff space. Let K be a compact subset of X and $x \in X$ with $x \notin K$. Since X is Hausdorff, K is closed. Now, since X is completely regular,there exists a continuous function $f: X \to [0,1]$ such that f(x)=0 and f(K)=1.Therefore X is strictly pseudo-regular.

Theorem 2.3: A topological space X is strictly pseudo-regular iffor each $x \in X$ and any compact set K not containing x, there exists an open set H of X such that $x \in H \subseteq \overline{H} \subseteq K^c$

Proof: Let X be a strictly pseudo-regular space and let K be compact in X. Let $x \notin K$ i.e., $x \in K^c$. Since X is strictly pseudo-regular,there exists a continuous function $f: X \to [0,1]$ such that f(x)=0 and f(K)=1. Let $a, b \in [0,1]$ and $a < b$. Then [0,a) and (b,1] are two disjoint open sets of [0,1]. Since f is continuous $f^{-1}([0,a))$ and $f^{-1}((b,1])$ are two disjoint open sets of X and obviously $x \in f^{-1}([0,a))$ and $K \subseteq f^{-1}((b,1])$. Let $U = f^{-1}([0,a))$ and $V = f^{-1}((b,1])$. Then $x \in U, K \subseteq V$ and $U \cap V = \phi$. Then $U \subseteq V^c \subseteq K^c$. So $\overline{U} \subseteq \overline{V^c} = V^c \subseteq K^c$. Writing U=H, we have $x \in H \subseteq \overline{H} \subseteq K^c$.

Theorem 2.4: Any subspace of a strictly pseudo-regular space is strictly pseudo-regular.

Proof: Let X be a strictly pseudo-regular space and $Y \subseteq X$. Let $y \in Y$ and K be a compact subset of Y such that $y \notin K$. Since $y \in Y$, so $y \in X$ and since K is compact in Y, so K is compact in X. Since X is strictly pseudo-regular,there exists a continuous function $f: X \to [0,1]$ such that f(y)=0 and f(K)=1. Therefore the restriction function $\overline{f}$ of f is a continuous function $\overline{f}: Y \to [0,1]$ such that f(y)=0 and f(K)=1. Hence Y is strictly pseudo-regular.

Corollary 2.1: Let X be a topological space and A, B are two strictly pseudo-regular subspace of X. Then $A \cap B$ is strictly pseudo-regular.

Proof: Since $A \cap B$ being a subspace of both A and B, $A \cap B$ is strictly pseudo-regular by the above theorem.

Theorem 2.5: Every strictly pseudo-regular space is Hausdorff.

Proof: Let X be a strictly pseudo-regular space. Let $x, y \in X$ with $x \neq y$. Then{x} is a compact set and $y \notin \{x\}$. Since X is strictly pseudo-regular, there exists a continuous function $f: X \to [0,1]$ such that f(y)=0 and f({x})=1. Let $a, b \in [0,1]$ and $a < b$. Then [0,a) and (b,1] are two disjoint open sets of [0,1]. Since f is continuous, f^{-1} ([0,a)) and f^{-1} ((b,1]) are two disjoint open sets of X and obviously $y \in f^{-1}$ ([0,a)) and {x}$\subseteq f^{-1}$ ((b,1]) i.e., $x \in f^{-1}$ ((b,1]). Therefore X is Hausdorff.

Theorem 2.6: Every strictly pseudo-regular space is pseudo regular.

Proof: Let X be a strictly pseudo-regular space. Let K be a compact subset of X and $x \in X$ with $x \notin K$. Since X is strictly pseudo-regular, there exists a continuous function $f: X \to [0,1]$ such that f(x)=0 and f(K)=1. Let $a, b \in [0,1]$ and $a < b$. Then [0,a) and (b,1] are two disjoint open sets of [0,1]. Since f is continuous, f^{-1} ([0,a)) and f^{-1} ((b,1]) are two disjoint open sets of X and obviously $x \in f^{-1}$ ([0,a)) andK$\subseteq f^{-1}$ ((b,1]) .Therefore X is pseudo regular.

We shall now definestrictly pseudo-normal spaces as a class of specialized pseudo normal spaces (see [17]) and proceed to study them.

III. STRICTLY PSEUDO-NORMAL SPACES

Definition 3.1: A topological space X will be called**strictly pseudo-normal** if for each pair of disjoint compact subsets K_1, K_2 of X, there exists a continuous function $f: X \to [0,1]$ such that f(K_1)=0 and f(K_2)=1.

Example 3.1: Let K_1, K_2 be two disjoint compact subsets of ℝ. Since ℝ isHausdorff, K_1, K_2 are also closed and since ℝ is completely normal, there exists a continuous function $f: X \to [0,1]$ such that f(K_1)=0 and f(K_2)=1. Thus ℝ is strictly pseudo-normal.

Theorem 3.1: Every strictly pseudo-normal compact space is completely normal.

Proof: Let X be compact and strictly pseudo-normal. Let K_1, K_2 be two disjoint closed subsets of X. Since X is compact, K_1, K_2 are also compact. Again, since X is strictly pseudo-normal,there exists a continuous function $f: X \to [0,1]$ such that f(K_1)=0 and f(K_2)=1.Therefore X is completely normal.

Theorem 3.2: Every completely normal Hausdorffspace is strictly pseudo-normal.

Proof: Let X be Hausdorff and completely normal. Let K_1, K_2 be two disjoint compact subsets of X. Since X is Hausdorff, K_1, K_2 are closed. Again, since X is completely normal, there exists a continuous function $f: X \to [0,1]$ such thatf(K_1)=0 and f(K_2)=1. Therefore X is strictly pseudo-normal.

Theorem 3.3: A topological space X is strictly pseudo-normal if each pair of disjoint compact sets K_1 and K_2, there existsan open set U such that $K_1 \subseteq U \subset \overline{U} \subseteq K_2^{\,c}$.

Proof: Let X be a strictly pseudo-normal space and K_1, K_2 be two compact subsets of X and $K_1 \cap K_2 = \phi$. Then $K_1 \subseteq K_2^{\,c}$. Since X is strictly pseudo-normal,there exists a continuous function $f: X \to [0,1]$ such that f(K_1)=0 and f(K_2)=1. Let $a, b \in [0,1]$ and $a < b$. Then [0,a) and (b,1] are two disjoint open sets of [0,1]. Since f is continuous f^{-1} ([0,a)) and f^{-1} ((b,1]) are two disjoint open sets of X and obviously $K_1 \subseteq f^{-1}$ ([0,a)) and $K_2 \subseteq f^{-1}$ ((b,1]). Let U = f^{-1} ([0,a)) and V = f^{-1} ((b,1]) . Then $K_1 \subseteq U$, $K_2 \subseteq V$ and $U \cap V = \phi$ .Then $U \subseteq V^c \subseteq K_2^{\,c}$. So $\overline{U} \subseteq \overline{V^c} = V^c \subseteq K_2^{\,c}$.Hence we have $K_1 \subseteq U \subset \overline{U} \subseteq K_2^{\,c}$.

Although a subspace of a normal space need not be normal (see [1], p.109), we have the following theorem:

Theorem 3.4: Every subspace of a strictly pseudo-normal space is strictly pseudo-normal.

Proof: Let X be a strictly pseudo-normal space and $Y \subseteq X$. Let K_1 and K_2 be two disjoint compact subsets of Y. Since K_1 and K_2 are compact in Y, these are compact in X too. Since X is strictly pseudo-normal,there exists a continuous function $f: X \to [0,1]$ such that f(K_1)=0 and f(K_2)=1. Therefore the restriction function $\overline{f}$ of f is a continuous function $\overline{f}: Y \to [0,1]$ such that f(K_1)=0 andf(K_2)=1. Hence Y is strictly pseudo-normal.

Theorem 3.5: Every strictly pseudo-normal space is Hausdorff.

Proof: Let X be a strictly pseudo-normalspace. Let $x, y \in X$ with $x \neq y$. Then{x} and {y}are two disjoint compact subsets of X. Since X is strictly pseudo-normal, there exists a continuous function $f: X \to [0,1]$ such that f({x})=0 and f({y})=1. Let $a, b \in [0,1]$ and $a < b$. Then [0,a) and (b,1] are two disjoint open sets of [0,1]. Sincef is continuous, f^{-1} ([0,a)) and f^{-1} ((b,1]) are two disjoint open sets of X and obviously $\{x\} \subseteq f^{-1}$ ([0,a)) i.e., $x \in f^{-1}$ ([0,a)) and {y}$\subseteq f^{-1}$ ((b,1]) i.e., $y \in f^{-1}$ ((b,1]) . Therefore X is Hausdorff.

Theorem 3.6: Every strictly pseudo-normal space is pseudonormal.

Proof: Let X be a strictly pseudo-normal space. Let K_1 and K_2 be two disjoint compact subsets of X. Since X is strictly pseudo-normal, there exists a continuous function $f : X \to [0,1]$ such that f(K_1)=0 and f(K_2)=1. Let $a,b \in [0,1]$ and $a < b$. Then [0,a) and (b,1] are two disjoint open sets of [0,1]. Since f is continuous, f^{-1} ([0,a)) and f^{-1} ((b,1]) are two disjoint open sets of X and $K_1 \subseteq f^{-1}$ ([0,a)), $K_2 \subseteq f^{-1}$ ((b,1]) .Therefore X is pseudo normal.

Theorem 3.7: Every strictly pseudo-normal space is strictly pseudo-regular.

Proof: Let X be a strictly pseudo-normal space. Let K be a compact subset of X and let $x \in X$ such that $x \notin K$. Therefore $\{x\}$ and K are disjoint compact subsets of X. Since X is strictly pseudo-normal, there exists a continuous function $f : X \to [0,1]$ such that f({x})=0 and f(K)=1 i.e., f(x)=0 and f(K)=1. Hence X is strictly pseudo-regular.

Theorem 3.8: Every metric space is both strictly pseudo-regular and strictly pseudo-normal.

Proof: Since every strictly pseudo-regular, strictly pseudo-normal spaces is pseudo-regular, pseudo-normaland since every metric space is both pseudo regular and pseudo normal, therefore it is both strictly pseudo-regular and strictly pseudo-normal.

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