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Applications of Laplace Transforms in Engineering and Economics

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Abstract: Laplace transform is a very powerful mathematical tool applied in various areas of engineering and science. With the increasing complexity of engineering problems, Laplace transforms help in solving complex problems with a very simple approach just like the applications of transfer functions to solve ordinary differential equations. This paper will discuss the applications of Laplace transforms in the area of mechanical followed by the application to civil. A more real time application on finance is also discussed.

Keywords: *Laplace Transform: Beam-Column: Present Discounted Value: Cash Flow.*

I. INTRODUCTION

Laplace transform methods have a key role to play in the modern approach to the analysis and design of engineering systems. The stimulus for developing these methods was the pioneering work of the English electrical engineer Oliver Heaviside (1850–1925) in developing a method for the systematic solution of ordinary differential equations with constant coefficients, as it enables them to solve linear differential equations with given initial conditions by using algebraic methods. The concepts of Laplace transform are applied in area of science and technology such as electric analysis, communication engineering, control engineering, linear system analysis, statistics optics and quantum physics etc. In solving problems relating to their fields, one usually encounters problems on time invariant, differential equations, time frequency domain for non periodic wave forms. This paper provides the reader to know about the fundamentals of Laplace transform and gain an understanding of some of the very important and basic applications to engineering field and economics problems.

Keywords: *Laplace Transform: Beam-Column: Present Discounted Value: Cash Flow.*

II. PROPERTIES OF LAPLACE TRANSFORM

Some of the important properties of Laplace transform which will be used in its applications are discussed below.

A. Linearity

The Laplace transform of the sum, or difference, of two signals in time domain is equal to the sum, or difference, of the transforms of each signals, that is,

$$L\left[C_1 f(t)+C_2 g(t)\right]=C_1 L\left[f(t)\right]+C_2 L\left[g(t)\right]$$

B. Differentiation

If the function $f(t)$ is piecewise continuous so that it has continuous derivative $f^{n-1}(t)$ of order $n-1$ and a sectionally continuous derivative $f^{n}(t)$ in every finite interval $[0,\infty]$, then

$$L\left[f^{n}(t)\right]=s^{n}L\left[f(t)\right]-s^{n-1}f(0)-s^{n-2}f^{1}(0)--------f(0)$$

C. Laplace transform of Unit step signal

$$L\left[u(t-a)\right]=\frac{e^{-as}}{s}$$

D. Laplace transform of Impulse signal

$$L\left[\delta(t-a)\right]=e^{-as}$$

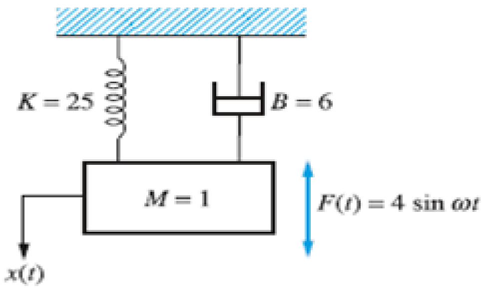
E. Second shifting theorem

$$L\left[f(t-a)u(t-a)\right]=e^{-as}L\left[f(t)\right]$$

III. APPLICATION IN MECHANICAL ENGINEERING

Vibrating Mechanical Systems: In examining the suspension system of the car the important elements in the system are the mass of the car and the springs and damper used to connect to the body of the car to the suspension links**.** Mechanical translational systems may be used to model many situations, and involve three basic elements: **masses** (having mass *M*, measured in kg), **springs** (having spring stiffness *K*, measured in Nm^{-1}) and **dampers** (having damping coefficient *B*, measured in Nsm^{-1}). The associated variables are **displacement** $x(t)$ (measured in m) and **force** $F(t)$ (measured in N).

Consider Mass–spring–damper system



Mass–spring–damper system can be modeled using Newton's and Hooke's law. Therefore the differential equation representing to the above system is given by

$$\frac{d^2x}{dt^2}+6\frac{dx}{dt}+25x=4\sin\omega t \qquad (1)$$

Taking Laplace transforms throughout in (1) gives

$$L\left[x''(t)\right]+6L\left[x'(t)\right]+25L\left[x(t)\right]=L\left[4\sin\omega t\right]$$

Incorporating properties of Laplace transform, we get

$$\left(s^2+6s+25\right)L\left[x(t)\right]=\left[sx(0)+x'(0)+6x(0)\right]+\frac{4\omega}{\left(s^2+\omega^2\right)}$$

And we take the initial conditions $x(0)=0=x'(0)$.

$$L\left[x(t)\right]=\frac{4\omega}{\left(s^2+\omega^2\right)\left(s^2+6s+25\right)}$$

which, on resolving into partial fractions $\left(\text{with } \omega=2\right)$, leads to

$$L\left[x(t)\right]=\frac{8}{\left(s^2+4\right)\left(s^2+6s+25\right)}=\frac{As+B}{\left(s^2+4\right)}+\frac{Cs+D}{\left(s^2+6s+25\right)}$$

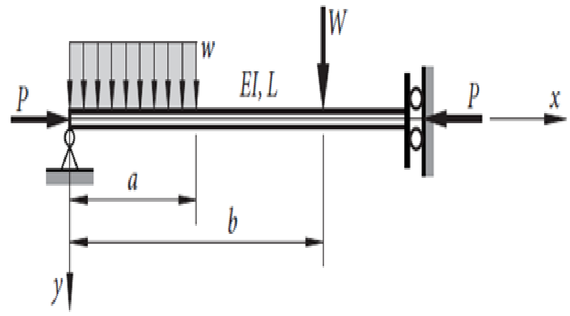
Taking inverse Laplace transforms gives the required response

$$x(t)=\frac{4}{195}\left(7\sin 2t-4\cos 2t\right)+\frac{2}{195}e^{-3t}\left(8\cos 4t-\sin 4t\right).$$

IV. APPLICATION IN CIVIL ENGINEERING

A. Beam-Column Problem

When the loading is non-uniform, the use of Laplace transform methods has a distinct advantage, since by making use of Heaviside unit functions and impulse functions. Figure illustrates a uniform beam of length l, freely supported at both ends, bending under uniformly distributed weight W.



Our aim is to determine the transverse deflection $y(x)$ of the beam.

From the elementary theory of beams, we have

$$EI\frac{d^4y}{dx^4}+P\frac{d^2y}{dx^2}=W(x) \qquad (2)$$

where $W(x)$ is the transverse force per unit length, with a downwards force taken to be positive, and EI is the flexural rigidity of the beam (E is Young's modulus of elasticity and I is the moment of inertia of the beam about its central axis). It is assumed that the beam has uniform elastic properties and a uniform cross-section over its length, so that both E and I are taken to be constants.

Using the Heaviside step function and the Dirac delta function, the force function $W(x)$ can be expresses as

$$W(x)=w\left[1-u(x-a)\right]+W\left[\delta(x-b)\right].$$

Therefore equation (1) become

$$EI\frac{d^4y}{dx^4}+P\frac{d^2y}{dx^2}=w\left[1-u(x-a)\right]+W\left[\delta(x-b)\right].$$

$$\frac{d^4y}{dx^4}+\beta^2\frac{d^2y}{dx^2}=\hat{w}\left[1-u(x-a)\right]+\hat{W}\left[\delta(x-b)\right].$$

Where $\beta^2=\frac{P}{EI}$, $\hat{w}=\frac{w}{EI}$ and $\hat{W}=\frac{W}{EI}$.

Since the left end is a hinge support and the right end is a sliding support, the boundary conditions are

$$\text{at } x=0 \qquad \text{deflection}=0 \;\Rightarrow\; y(0)=0.$$

$$\text{bending moment}=0 \;\Rightarrow\; y''(0)=0.$$

$$\text{at } x=L \qquad \text{slope}=0 \;\Rightarrow\; y'(L)=0.$$

$$\text{shear force}=0 \;\Rightarrow\; y'''(L)=0.$$

Applying the Laplace transform, we get

$$L\left[y''''(x)\right]+a^2L\left[y''(x)\right]=\hat{w}\,L\left[1-u(x-a)\right]+\hat{W}\,L\left[\delta(x-b)\right]$$

Incorporating properties and Laplace transform of impulse and step function, we get

$$\{s^4L[y(x)]-sy'(0)-y'''(0)\}+a^2\{s^2L[y(x)]-y'(0)\}=\hat{w}\frac{[1-e^{-as}]}{s}+\hat{W}e^{-bs}$$

$$s^2(s^2+a^2)L[y(x)]=s^2y'(0)+y'''(0)+a^2y'(0)+\hat{w}\frac{[1-e^{-as}]}{s}+\hat{W}e^{-bs}$$

$$L[y(x)]=\frac{y'(0)}{(s^2+a^2)}+\frac{[y'''(0)+a^2y'(0)]}{s^2(s^2+a^2)}+\hat{W}\frac{e^{-bs}}{s^2(s^2+a^2)}+\hat{w}\frac{[1-e^{-as}]}{s^3(s^2+a^2)}$$

Taking inverse transforms, making use of the second shift theorem, gives the deflection $y(x)$ as

$$L[y(x)]=\left\{\begin{array}{l}\frac{y'(0)}{a}\sin at+[y'''(0)+a^2y'(0)]\frac{1}{a^2}\left(t-\frac{1}{a}\sin at\right)\\ +\hat{W}\left(\frac{t-b}{a^2}-\frac{1}{a^3}\sin a(t-b)\right)u(t-b)\\ +\hat{w}\left[\frac{1}{2a^4}\left(a^2t^2+2(\cos at-1)\right)\right]\\ +\hat{w}\left[\frac{1}{2a^4}\left(a^2(t-a)^2+2(\cos a(t-a)-1)\right)\right]\end{array}\right\}$$

To obtain the value of the undetermined constants $y'(0)$ and $y'''(0)$, we employ the unused boundary conditions at $x=L$ namely $y'(L)=0$ and $y'''(L)=0$.

V. APPLICATION IN ECONOMIC PROBLEMS

A. First we obtain the Relation between present value and Laplace transform

Consider case of investment project. For various alternatives one wishes to calculate the present value of series of cash receipts and transactions. The Present value of a series of payments given by,

$$[PV]_t=\sum_{t=1}^{T}\frac{C(t)}{(1+r)^t} \qquad (3)$$

Where, $[PV]_t$ = present discounted value at time t

$C(t)$ = Cash flow

r = rate of discount and t = period

Here we assume the present value with continuous compounding. It is the current value of a stream of cash flows. In other words, it is the amount that we would be willing to pay today in order to receive a cash flow or a series of them in the future.

Now by using an exponential series we write equation (3) as

$$[PV]_t=\sum_{t=1}^{T}C(t)e^{-rt} \qquad (4)$$

In the continuous time case replacing summation to an integral, equation (4) Can be written as

$$[PV]_r=\int_0^T C(t)e^{-rt}\,dt \qquad (5)$$

Again here T is some finite quantity. So if we consider as T→∞, equation (5) Becomes

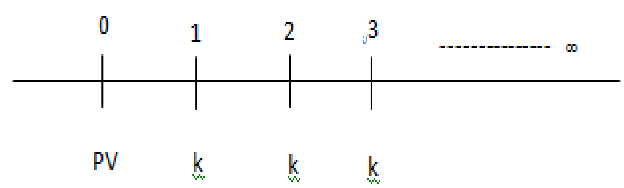
$$[PV]_r=\int_0^\infty e^{-rt}C(t)\,dt \qquad (6)$$

By using definition of Laplace transform , equation (6) can be written as

$$[PV]_r=L[C(t)]$$

B. Laplace transform and present value for cash flows Using present value

Consider the case of constant cash payment K made at the end of each year at interest rate r as shown in the following time line,



Here the cash flows is continuous forever. Therefore the present value is given by an infinite geometric series:

$$PV=\frac{K}{(1+r)}+\frac{K}{(1+r)^2}+\frac{K}{(1+r)^3}+-------- \quad (7)$$

Dividing both sides by $(1+r)$ *we get,*

$$\frac{PV}{(1+r)}=\frac{K}{(1+r)^2}+\frac{K}{(1+r)^3}+\frac{K}{(1+r)^4}+-------- \quad (8)$$

subtracting equation (8) *from* (7) *we get*

$$PV\left[\frac{r}{(1+r)}\right]=\frac{K}{(1+r)}$$

On solving we get the Present value

$$\Rightarrow PV=\frac{K}{r}=L[K]$$

Using Laplace transform equation: If the cash flow is constant say K then the Present discounted value of a stream at interest rate r is given by

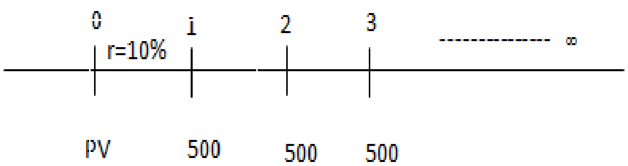
$$L[K] = K\int_0^{\infty} e^{-rt}dt = \frac{K}{r}$$

This is the same formula as above.

C. Example

An insurance company has just launched a security that will pay Rs.500 indefinitely, starting the first payment next year. How much should this security be worth today if the appropriate return is 10% ?

We solve this example by using the time line,



$$\Rightarrow PV = \frac{K}{r} = \frac{500}{0.1} = Rs\,5000.$$

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